Pergamon

PII: S0021-8928(98)00105-1

J. Appl. Maths Mechs, Vol. 62, No. 5, pp. 823–826, 1998
© 1999 Elsevier Science Ltd
All rights reserved. Printed in Great Britain
0021-8928/98/\$—see front matter

THE ESTIMATE OF THE TIME OF MOTION OF CERTAIN DYNAMICAL SYSTEMS†

A. P. BLINOV

Moscow

(*Received* 29 *January* 1998)

An estimate of the time necessary for the phase points of a dynamical system to reach a specified finite domain, containing an asymptotically stable solution, from any initial position belonging to the specified domain is obtained with the sole assumption that the derivative of the Lyapunov function for autonomous second-order systems and for certain higher-order systems has a negative sign. © 1999 Elsevier Science Ltd. All rights reserved.

In [1] an estimate of the upper limit of the time was obtained in the case when the Lyapunov function has a strictly negative derivative.

1. Suppose the origin of the coordinates (the point O) is an asymptotically stable solution of the system of equations

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y) \tag{1.1}$$

for which a positive-definite Lyapunov function $V(x, y)$ is known such that $\dot{V}(x, y) \leqslant 0$ along the solutions.

We shall now estimate the time of motion of any phase point lying on the boundary of a domain G_0, which is defined by the equation $V(x, y) = C_0$ up to the boundary of a specified domain G_*, which is defined by the equation $V(x, y) = C_*$.

We will denote the lines of the family $V(x, y) = C$, corresponding to the parameter C, and, also, their lengths by $l(C)$.

Suppose the origin of coordinates is a singular point of the stable-focus type, and the phase point which coincides with the point A_0 (Fig. 1) at the time $t = 0$ makes one complete circuit around the origin of coordinates after a time t_0 along a trajectory of length γ, where this trajectory is not closed. (By a "complete circuit", we mean a motion in which the start and the finish of the trajectory γ lie on the same line of the family $L(\beta)$ which is orthogonal to $l(C)$.)

Suppose $l(C)$ is a family of smooth convex curves when $C \leqslant C_0$ and the trajectory γ is convex with respect to the point O. Then

$$\gamma \leqslant l(C_0) \tag{1.2}$$

This can be shown to be so directly if $l(C_0)$ is a circle of radius ρ_0 and γ is a logarithmic spiral and the difference between the increments in the arcs of the circle and the spiral is equal to

$$\rho_0(1 - \cos\alpha_0)\,\varphi + O(\varphi^2),$$

where α_0 is the acute angle between the tangents to the circle and the spiral, and φ is the polar angle of the radius vector, measured from the point of intersection of the curves in the direction of the convolution of the spiral.

In the general case, suppose Γ is a line (or several lines) in which $\dot{V}(x, y) = 0$ and Γ_1 and Γ_2 are lines which are obtained, for example, from Γ by rotation through a small angle (or are represented in the form $y = (1 \pm \delta)\Phi(x)$, where $y = \Phi(x)$ is the equation of the line Γ and δ is a small positive number (Fig. 1)).

Suppose B is the set of sectors, included between the lines Γ_1 and Γ_2 and containing Γ. The arcs of γ and $l(C)$ touch on the line Γ. Hence, within B, they are equal to an accuracy of $o(\delta)$. Outside the domains B and G_*, the trajectory γ intersects the lines $l(C)$ at an angle $\alpha_C \geqslant \alpha_0 > 0$.

Suppose O_0 is the centre of curvature for $l(C_0)$ at a point $A_0 \notin B$. In a small circle K_0 of radius δ_0 with its centre at this point, the family of lines $l(C)$ can be approximated by a family of circles with a common centre O_0 and the line γ is a logarithmic spiral to an accuracy $o(\delta_0)$. A ray traced from the point O_0 through a point $A_1 \notin \gamma \cap K_0$ intersects the arc $A_0A'_0$, which must be greater than the arc A_0A_1 of the line γ, in $A_1 \notin B$, we determine the centre of curvature O_1 of the line $l(C_1) \ni A_1$ and a circle K_1 of radius δ_1, a point $A_2 \notin \gamma \cap K_2$ and the ray O_1A_2. In $l(C_1)$, this ray intersects the arc $A_1A'_1$, which must be smaller than the arc $A_1A_2 \in \gamma$ since the latter arc is also approximated by an element of a logarithmic spiral.

†*Prikl. Mat. Mekh.* Vol. 62, No. 5, pp. 888–892, 1998.

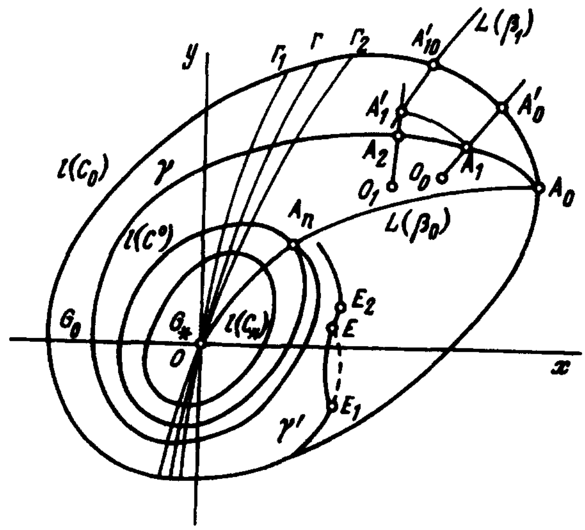

Fig. 1.

Suppose that $L(\beta_1)$, a line of the family $L(\beta)$ which passes through the point A'_1, intersects the line $l(C_0)$ at the point A'_{10}. By virtue of the convexity of $l(C)$, the arc $A'_0A'_{10}$ is greater than the arc $A_1A'_1$. If $A_1 \in B$, the intersection of γ with Γ_1 can be taken as the point A_2.

So, on continuing to move along γ, the line $l(C_0)$ is subdivided into the sum of the arcs $A_0A'_0, A'_0A'_{10}, \ldots$, each of which, to an accuracy of $o(\delta_k)$, is greater than the corresponding element of the line γ. On taking the limit as $n \in \infty$ $(\max(\delta, \delta_0, \delta_1, \ldots, \delta_n) \to 0)$, we obtain inequality (1.2).

Similar reasoning leads to the inequality

$$l(C_*) \leq \gamma \tag{1.3}$$

The assumption made above regarding the convexity of the curve γ can be discarded if γ intersects all the lines $l(C)$ at an acute angle $(\alpha_C < \pi/2)$.

In fact, suppose a certain trajectory γ' (Fig. 1) is concave between the points E_1 and E_2. By assumption, the angle $\alpha_1 > \alpha_0$ at the point E_1. A ray can then be drawn from the point E_1 at an angle of $\alpha_0/2$ to the tangent to γ' which intersects the arc E_1E_2 at a certain point E. On specularly reflecting the arc E_1E with respect to the above-mentioned arc, we obtain a convex arc E_1E (the dashed curve in Fig. 1) which intersects the line $l(C)$ at an acute angle. Consequently, the arguments presented above can be applied to this arc. The procedure which has been described can be repeated at the point E, if this point does not lie in the sector B. Hence, after a finite number of steps, we arrive at the point E_2.

We will now show that the condition $\alpha_C < \pi/2$ is satisfied in the case of one-degree-of-freedom mechanical systems.

The motion of such a system is described by the equations

$$\dot{x} = y, \quad \dot{y} = -b(x, y) - c(x)$$

where $b(x,y) \geq 0$ and $c(x) \geq 0$ are dissipative and conservative forces respectively. The angle α_C between the phase velocity vector υ and the vector υ_0 tangent to the line $l(C)$, which, in this case, is also described by the equation that has been given when $b(x,y)$ ∀ 0, remains acute while the scalar product $\upsilon\upsilon_0 \geq 0$ or $y^2 + c^2(x) + c(x)b(x,y) \geq 0$, which is certainly satisfied. The requirement $\upsilon\upsilon_0 \geq 0$ remains for equations of general form.

After a time t_0, the parameter C_0 decreases by an amount ΔC $(\Delta C > 0)$. On the basis of inequalities (1.2) and (1.3) it is possible to formulate the following inequalities

$$l(C^0)/W_M < t_0 < l(C_0)/W_m \tag{1.4}$$

$$C^0 = C_0 - \Delta C, \quad W_m = \inf_{G_{00}} W, \quad W_M = \sup_{G_{00}} W, \quad W = \sqrt{X^2 + Y^2}$$

$(G_{00} = G_0 \backslash G^0$ is the annular domain between the lines $l(C_0)$ and $l(C^0))$.

The value W_M is often attained on the line $l(C_0)$. For some simplication, this case will be considered further.

The time of motion t_B of a phase point through the sectors B can be estimated by the inequality $t_B \leq l_B/W_B$, where l_B is the sum of the lengths of the portions of the line $l(C_0)$ lying in B and $W_B = \inf W$ in the domain $B_0 = B \cap G_{00}$.

Then

$$\Delta C = -\int_0^{t_0} \dot{V} dt \geqslant \mu(t_0 - t_B), \quad \mu = \inf_{B_0} |\dot{V}|$$

or, using (1.4)

$$\Delta C \geqslant \mu[l(C^0)/W_M - l_B/W_B] \tag{1.5}$$

The solution of this inequality depends on the choice of the width of the band B, that is, the right-hand side can be maximized by the choice of l_B. However, in the first step, the choice of l_B must be such that the right-hand side of (1.5) is negative for any $\Delta C \leqslant C_0 - C_*$. For this purpose, it is sufficient to take $l_B = l_{B^*} = l(C_*)W_{m^*}/W_M$, $W_{m^*} =$ inf W in the domain $G_{0^*} = G_0 \backslash G_*$.

When $l_{B^*} \leqslant l_B \leqslant l(C_0)$, a certain (or, better, the maximum) value of ΔC, which satisfies the inequality (1.5), is denoted by ΔC_*.

Remark. The right-hand side of (1.5) in the case of fixed l_{B^*} increases, starting from zero, and the left-hand side of $\Delta C = C_0 - C^0$ decreases as C^0 increases from C_* to C_0. A value $C^0 = C^{0*}$ is therefore found for which inequality (1.5) becomes an equality. The corresponding value for ΔC can be taken as ΔC_* but ΔC can still be maximized by a new choice of l_B in the interval $[l_{B^*}, l_{B0}]$, $l_{B0} = l(C^{0*})W_{m0}/W_M$, $W_{m0} =$ inf$W$ in the domain between the lines $l(C_0)$ and $l(C_{0^*})$ (instead of fixing $l_B = L_{B^*}$ as was done in the first step).

The following estimate can be given for the time t_0

$$\max[\Delta C_*/M;\ l(C_*)/W_M] \leqslant t_0 \leqslant \min[(C_0 - C_*)/\mu + l_B/W_B;\ l(C_0)/W_{m^*}] \tag{1.6}$$

$M = \sup[\Delta C_*/M; l(C_*)/W_M]$ in the domain G_{0^*}.

Next, starting from the value of $C = C_0 - \Delta C_*$, we obtain an estimate of the time of motion of a phase point in the second loop and so on until it enters the domain G_*. (Hence, in estimating the time of motion of a phase point until it enters the domain G_*, an upper estimate of both the time of motion in the loop as well as the number of loops is made.)

The estimates which have been given are also suitable for an unstable focus or node if one is interested in the time of departure from the domain G_{0^*}.

Example. Suppose the motion of a pendulum with a dissipative and non-linear elastic coupling is given in dimensionless form by the equations

$$\dot{x} = y, \quad \dot{y} = -y - 2x^3$$

Here, $W = [y^2 + (y + 2x^3)^2]^{1/2}$, $V = x^4 + y^2)/2$, $\dot{V} = -y^2$ and the lines $V(x, y) = C$ are convex.

Suppose $C_0 = 10$, $C_* = 5$. Then, $l(C_*) = 16.8$ (graphical solution) and $W_{m^*} = W_B \geqslant 4.5$, $W_M = 18.9$, $l_{B^*} = 4$.

We take the strip between the straight lines $y = \pm 1$ as the sector B. Then, $\mu = 1$.

We now take $C^0 = 9$. Then, $l(C^0) = 20.8$, $W_B = 6.0$. In this case, the right-hand side of (1.5) is equal to 0.43 which is still smaller than $\Delta C = C_0 - C^0 = 1$. In the following step, we therefore put $C^0 = 9.5$. On calculating the right-hand side of (1.5) again, we obtain its value as 0.55, which enables us to take $\Delta C_* \leqslant 0.5$. Then, by (1.6), we obtain $0.9 \leqslant t_0 \leqslant 5.37$.

2. In the case of multidimensional dynamical systems under conditions where Corduneanu's theorem [1] for estimating T, the time of motion of a phase point until it enters the specified domain, applies, we can use well-known results [1, paragraphs a and b, p. 67]. In fact, from paragraph a in [1], we have

$$a(\|x\|) \leqslant u(t; t_0, V(t_0, x_0))$$

(t_0 is the initial instant of time). Fixing the finite domain to be reached as $\|x\| \leqslant b$, we obtain

$$a(b) \leqslant u(t_0 + T,\ t_0,\ V(t_0, x_0)) \tag{2.1}$$

An estimate for T also follows from the last inequality.

Example 1 (an analogue of the example from [1, p. 68]). Consider the system

$$\dot{x} = (-Et + A(t, x))x$$

where E is the identity matrix and $A(t, x)$ is a skew symmetric matrix. Suppose $V = x_1^2 + x_2^2 + \ldots + x_n^2$. Then, $\dot{V} = 2x_1(-tx_1) + 2x_2(-tx_2) + \ldots = -2Vt$.

The solution $u = 0$ of the scalar comparison equation $\dot{u} = 2ut$ (here, $\omega(t, u) = -2ut$ and the inequality $\dot{V} \leqslant \omega(t, V)$ is satisfied) is asymptotically stable, since the general solution $u = C \exp(-t^2)$.

We now put $t_0 = 0$; $\| x_0 \| = e^2$; $b = 1$. Since $u(t_0) = u(t_0; t_0, V(t_0, x_0))$, then $u(t_0) = V(t_0, x_0) = \| x_0 \|^2 = e^4$. On the other hand, $u(t_0) = C \exp(-t_0^2)$, whence we have $C = e^4$ and $u(T) = \exp(4 - T^2)$.

On taking account of the fact that, in the given case, $a(\| x \|) = V = \| x \|^2$, from inequality (2.1) we obtain

$$T \leqslant 2$$

Since the function $\dot{V}$ is negative definite in this example, an estimate of T can be obtained from the inequality

$$\Delta V = -\int_0^T \dot{V} dt = 2\int_0^T V t dt \geqslant \inf_{G_{0*}} V \cdot T^2 \tag{2.2}$$

$$\Delta V = e^4 - 1, \ \inf_{G_{0*}} V = 2; \ T \leqslant 5$$

Example 2. For the system

$$\dot{x} = (-E \sin^2 t + A(t, x))x$$

we have that $\dot{V} = -2V \sin^2 t$, and inequality (2.2) cannot be used. However, using (2.1), we obtain $T \leqslant 4.5$.

Under the conditions where Matrosov's theorem [1, pp. 58 and 59] is applicable, it is also possible to obtain an estimate of T using the basic estimates in the proof of this theorem. In fact, if we take the domain $C(\| x \|) \geqslant \eta$ to be the domain G_* (for the meanings of the new symbols used below, see [1]), it is possible to write $T \leqslant (K + 1)(\tau_2 - \tau_1)$, where $\tau_2 - \tau_1 = 2L/\eta$ and the quantity K is determined from the inequality

$$C_0 - C_* \geqslant K \min [2lL/\eta, h\eta/(2A)]$$

I wish to thank V. S. Sergeyev for critical comments in discussing the papers.

REFERENCE

1. ROUCH, N., HABETS, P. AND LALOY, M., *Stability Theory by Lyapunov's Direct Method*. Springer, New York, 1977.

Translated by E.L.S.